

Last time Every isometry of $\mathbb{R}^n$, f , is of the form

$$f = t_a \rho$$

where $t_a(x) = x + a$, $a \in \mathbb{R}^n$ (translation by a)
 $\rho \in O_n$.

Lemma: Suppose $\rho \in O_n$, $a \in \mathbb{R}^n$. Then

$$\rho t_a \rho^{-1} = t_{\rho(a)}$$

Pf: Take $x \in \mathbb{R}^n$. Then

$$\rho t_a \rho^{-1}(x) = \rho(t_a(\rho^{-1}(x)))$$

$$= \rho(\rho^{-1}(x) + a)$$

$$= x + \rho(a)$$

$$= t_{\rho(a)}(x) \checkmark \quad \square$$

This implies $(t_a \rho)(t_b \rho')$
 $\parallel$

$$a, b \in \mathbb{R}^n \\ \rho, \rho' \in O_n$$

$$t_a(\rho t_b \rho'^{-1}) \rho \rho' = t_a t_{\rho(b)} \rho \rho' \\ = t_{a+\rho(b)} \rho \rho'$$

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We proved last time, if $f \in M_n$ then $f = t_a \rho$ some a, ρ .

Denote by $\pi: M_n \rightarrow O_n$ defined as $\pi(t_a \rho) = \rho$

Note: If $f \in M_n$, then in the decomposition $f = t_a \rho$, a and ρ are unique.

$$\text{I.e. } t_a \rho = t_{a'} \rho' \Rightarrow a = a', \rho = \rho'.$$

Propⁿ $\pi: M_n \rightarrow O_n$ is a surjective gp homom.

$$\begin{aligned} \text{Pf: } \pi((t_a \rho)(t_b \rho')) &= \pi(t_{a+\rho(b')} \rho \rho') = \rho \rho' \\ &= \pi(t_a \rho) \pi(t_b \rho'). \end{aligned}$$

• Surjectivity is immediate. $\square$

Note: This is similar to HW exercises when you considered

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mapsto A \in GL_n(\mathbb{R})$$

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and checked that it's a homom.

Propⁿ There is an injective gp homom. $\varphi: M_n \rightarrow GL_{n+1}(\mathbb{R})$

$$\text{defined as } \varphi(t_a \rho) = \begin{pmatrix} 1 & 0 \\ a & \rho \end{pmatrix}_{n+1}$$

$\underbrace{\quad\quad\quad}_1 \quad \underbrace{\quad\quad\quad}_n$

Pf: clearly injective.

$$\varphi(t_a p) \varphi(t_b p') = \left(\begin{array}{c|c} 1 & 0 \\ \hline a & p \end{array} \right) \left(\begin{array}{c|c} 1 & 0 \\ \hline b & p' \end{array} \right)$$

$$= \left(\begin{array}{c|c} 1 & 0 \\ \hline a + p(b) & p p' \end{array} \right)$$

$$= \varphi(t_{a+p(b)} p p')$$

$$= \varphi((t_a p)(t_b p')).$$

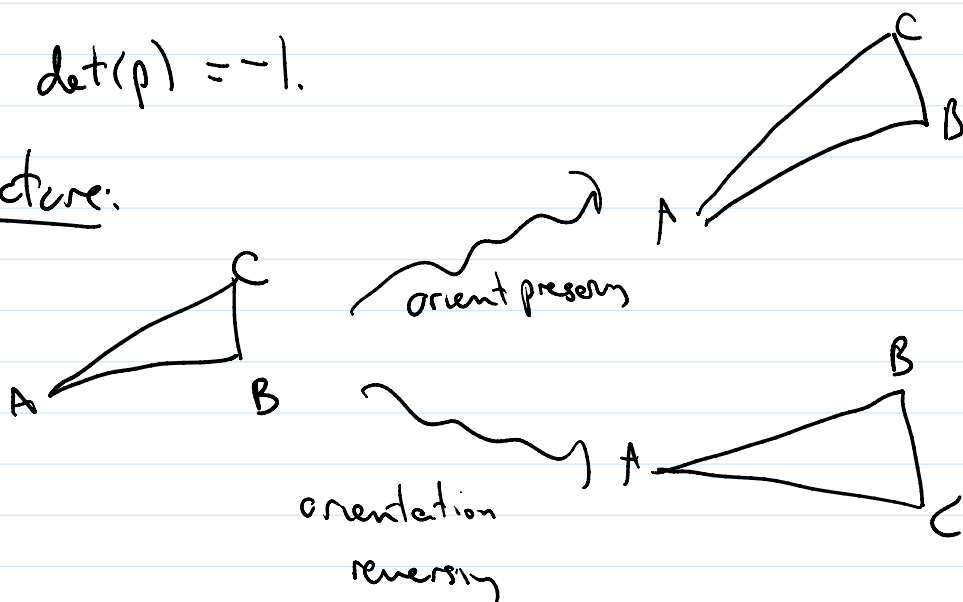
Defⁿ Suppose $f = t_a p \in M_n$, i.e. f is an isom. of $\mathbb{R}^n$.

Then f is said to be orientation-preserving if

$\det(p) = 1$. f is said to be orientation-reversing if

$\det(p) = -1$.

Picture:



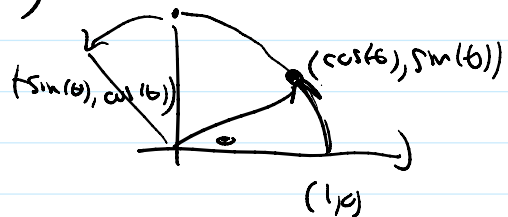
Notation for $\theta \in \mathbb{R}$ define

$$p_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

This is the matrix for rotation by θ in $\mathbb{R}^2$

b/c $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}$$



Propⁿ Every elt of $SO(2)$ is p_θ for some θ .

Pf: If $g \in SO(2)$ then $\det(g) = 1$ and ${}^t g g = \underline{1_2}$,

$$\Rightarrow g = {}^t g^{-1}$$

$$\text{If } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } {}^t g^{-1} = \frac{1}{\det(g)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

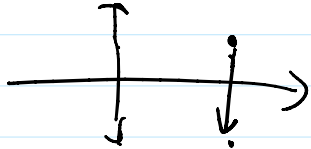
$$\Rightarrow d = a, \quad b = -c$$

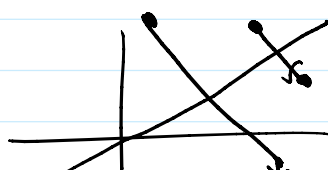
$$\Rightarrow g = \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \quad \det(g) = 1 = a^2 + c^2$$

$$\Rightarrow a = \cos(\theta), \quad c = \sin(\theta) \text{ for some } \theta \in \mathbb{R}.$$

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Propⁿ Every orientation-reversing element $p \in O(2)$ is a reflection across some line thru the origin.

$$r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightsquigarrow$$


$$r_\theta$$


Lemma: Set $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then

$$r p_\theta r^{-1} = r p_\theta r = p_{-\theta}$$

Pf: check:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

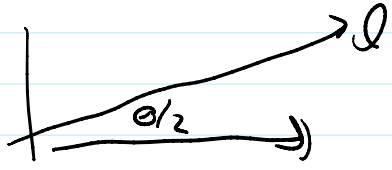
$$= \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = p_{-\theta}. \quad \square$$

Pf of Propⁿ $p = p_\theta r$ for some angle θ .

Claim: (1) $p_\theta r = p_{\theta/2} r p_{-\theta/2}$

(2) $p_{\theta/2} r p_{-\theta/2}$ is reflection across the line ℓ
 $\rightarrow 0$

(2) $P_{\theta/2} \circ P_{-\theta/2}$ is reflection across the line x



Pf of Claim: (1) Rearrange to obtain

$$P_{\theta/2} = P_{\theta/2} P_0 \stackrel{?}{=} r P_{\theta/2} r^{-1} \quad \text{and this follows from lemma.}$$

(2) Think geometrically

